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Equivalent head-wind statistics for variable tracks.

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(Meteorological Office)

AIR MINISTRY

METEOROLOGICAL RESEARCH COMMITTEE

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Summary

The frequency distribution of equivalent head-winds over an air route may be computed by statistical methods, using certain assumptions, from mean upper wind charts. For any given air route, however, economy of fuel may be achieved by flying the most favourable, from a wind point of view, of several fixed tracks. The statistical technique is extended to illustrate how the resulting frequency distribution of equivalent head-winds may be computed, and the relative merits of the different tracks assessed. Illustrations are given for the London-New York route, using two or three alternative tracks.

Introduction

The theory of equivalent head-winds has been outlined by Sawyer[1] and Crossley[2]. On the premise that seasonal upper winds at a point possess a normal, circular distribution, and that the form of correlation of wind with distance is known, they have shown that the frequency distribution of route equivalent head-winds can readily be computed from charts of mean vector wind and standard vector deviation.

Over long-distance routes, a considerable time-saving may be effected by flying the minimal flight path. For operational and other reasons, however, this is generally impracticable and a compromise method is to fly the most favourable of several fixed tracks. For planning purposes, the resulting frequency distribution of flight durations requires lengthy tabulations. For instance, an aircraft operator planning a new route has to make a detailed analysis of upper-air contour charts over the route in question over a period, preferably, of several years.

A statistical method is given below, enabling a fairly rapid computation to be made of the frequency distribution of equivalent head-winds, using the variable track technique. Though the theory is applicable to any number of tracks, it is too cumbersome to apply when more than three are considered. The basic assumptions made are largely the same as those which are at present used in computing equivalent head-wind statistics for single tracks and which have proved sufficiently accurate in practice.

Correlation of route winds

Durst[3] has found that the correlation of point winds with distance is largely independent of height, but varies to some extent with latitude and direction. In this paper, the form of correlation with distance is assumed constant, and the values used are applicable to temperate latitudes. Statistics of wind over single tracks computed in the Meteorological Office likewise assume constant correlation, but use separate curves for temperate and tropical regions.

The correlation of mean equivalent head-winds over two different routes can now be derived as follows:-

/Consider

Consider two parallel routes (Fig. 1) each of length S, and distance apart h. Let u(m) and v(n) be the departures of the equivalent head-winds from their long-period means at any points m and n, along the first and second routes respectively.

The departures, U and V, of the route mean equivalent head-winds from their long-period means are:

$$U = \frac{1}{S}\int_0^S u(m)\,dm \;; \quad V = \frac{1}{S}\int_0^S v(n)\,dn \qquad \ldots\ldots (1)$$

Let σ_T be the standard deviation of the route equivalent head-winds (for simplicity assumed the same for both routes).

Following Sawyer, we take σ^2, the variance of the point equivalent head-winds, to be constant, and equal to its mean value for the route. Sawyer has shown that:

$$\sigma_T^2 = \frac{2\sigma^2}{S^2}\int_0^S\int_0^\Delta R(t)\,dt\,d\Delta \qquad \ldots\ldots (2)$$

where R(t) is the correlation coefficient between winds at points distant t apart.

We require the correlation coefficient ρ between the mean winds on the two routes:

Now
$$\rho = \frac{\overline{UV}}{\sigma_T^2} \qquad \ldots\ldots (3)$$

and from (1)
$$UV = \frac{1}{S^2}\int_0^S\int_0^S u(m)\,v(n)\,dm\,dn$$

Hence, from (2) and (3)
$$\rho = \frac{\int_0^S\int_0^S \overline{u(m)\,v(n)}\;dm\,dn}{2\sigma^2\int_0^S\int_0^\Delta R(t)\,dt\,d\Delta}$$

But
$$\overline{u(m)\,v(n)} = \sigma^2 R(p)$$

where
$$p^2 = h^2 + (m-n)^2 \text{ as shown in fig. 1.}$$

Hence
$$\rho = \frac{\int_0^S\int_0^S R(p)\,dm\,dn}{2\int_0^S\int_0^\Delta R(t)\,dt\,d\Delta} \qquad \ldots\ldots (4)$$

Integrating,
$$\int\left[\int_0^t R(t)\,dt\right]dt = t\int_0^t R(t)\,dt - \int t\,R(t)\,dt$$

whence
$$\int_0^S\int_0^\Delta R(t)\,dt\,d\Delta = \int_0^S (S-t)\,R(t)\,dt$$

/Also

Also it is easily seen that

$$\int_0^S \int_0^S R(\rho)\, dm\, dn = 2\int_0^S (S-t)\, R(\rho)\, dt$$

where, now, $$\rho^2 = h^2 + t^2$$

Equation (4) therefore reduces to

$$\rho = \frac{\int_0^S (S-t)\, R(\rho)\, dt}{\int_0^S (S-t)\, R(t)\, dt}$$

where ρ is the correlation coefficient between the mean equivalent head-winds over two parallel routes, length S, separation h; R(t) is the correlation coefficient between winds at points distant t apart, and $p^2 = h^2 + t^2$.

Variation of route wind correlation

From the observed form of variation of R(t) with distance, it is clear that ρ will increase with S, becoming almost constant for route lengths greater than about 1,500 miles. An exponential decrease of point wind correlation with distance is probably the best assumption that can at present be made, and the curve of best fit to those given by Durst is

$$R(t) = \exp(-0.2\, t)$$

in temperate latitudes, where t is measured in hundreds of nautical miles.

The integral $$\int_0^S \exp\left(-0.2\sqrt{h^2+t^2}\right) dt$$

has been tabulated for different values of h and S, and the following are corresponding values of ρ.

Table 1. Correlation coefficients between mean equivalent head-winds over two parallel routes, length S, separation h

	Values of h (nautical miles)			
S(n.m.)	200	400	600	800
0	0.67	0.45	0.30	0.20
500	0.80	0.56	0.38	0.26
1000	0.83	0.61	0.43	0.30
2000	0.85	0.64	0.47	0.34
3000	0.86	0.66	0.49	0.36
4000	0.87	0.67	0.50	0.36

In the practical case, the mean distance apart of two non-parallel routes is used to obtain an approximation to ρ, the route lengths being approximately equal.

Wind distribution for two tracks

Let x and y be equivalent head-winds on the shorter and longer of two fixed tracks respectively, means $\bar{x}$, $\bar{y}$ and standard deviations σ_1, σ_2. Let a be the value by which the equivalent head-wind on the shorter route must exceed

/that

that on the longer for the flight durations to be equal, i.e. the longer route is flown only if $y < x - a$. a is a function of the route-lengths, aircraft's still air speed and wind velocity. Provided the aircraft's speed greatly exceeds that of the wind, the variation in a will be small. In the examples below, a has been taken as constant, though its exact value could be used if required.

We want the distribution of r, where r is the lesser of the two winds x and y + a, and is a 'compensated' equivalent head-wind, referring always to the shorter route length, whichever track is flown.

r will lie between r and r + dr if <u>either</u>

A) $r < x < r + dr$ <u>and</u> $r - a < y$

<u>or</u> B) $r - a < y < r - a + dr$ <u>and</u> $r < x$

i.e. P (r lies between r and r + dr) = P(A) + P(B)
where P denotes probability.

The bivariate normal frequency distribution of x, y, as given in most text-books is:

$$\phi(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)}\left(\frac{(x-\bar{x})^2}{\sigma_1^2} + \frac{(y-\bar{y})^2}{\sigma_2^2} - 2\rho\frac{(x-\bar{x})(y-\bar{y})}{\sigma_1\sigma_2}\right)\right]$$

and

$$P(A) = dr \int_{r-a}^{\infty} \phi(r, y)\, dy$$

$$P(B) = dr \int_{r}^{\infty} \phi(x, r-a)\, dx$$

Simplifying

$$P(A) = \frac{1}{\pi\sigma_1\sqrt{2}} \exp\left(-\frac{(r-\bar{x})^2}{2\sigma_1^2}\right) dr \int_{q_1}^{\infty} \exp(-z^2)\, dz$$

and

$$P(B) = \frac{1}{\pi\sigma_2\sqrt{2}} \exp\left(-\frac{(r-a-\bar{y})^2}{2\sigma_2^2}\right) dr \int_{q_2}^{\infty} \exp(-z^2)\, dz$$

where

$$q_1 = \frac{1}{\sqrt{2}\sqrt{1-\rho^2}}\left(\frac{r-a-\bar{y}}{\sigma_2} - \frac{\rho(r-\bar{x})}{\sigma_1}\right)$$

$$q_2 = \frac{1}{\sqrt{2}\sqrt{1-\rho^2}}\left(\frac{r-\bar{x}}{\sigma_1} - \frac{\rho(r-a-\bar{y})}{\sigma_2}\right)$$

The distribution may be computed by evaluating P(A) and P(B) for, say, 5-knot intervals, using tabulations of the normal error function.

<u>London- New York route - two tracks</u>

The distribution has been computed for the London - New York route using the following two tracks: (Fig. 2)

1. London - Fastnet - Torbay - New York (3008 nautical miles)

2. London - Malin - Goose Bay - New York (3055 nautical miles).

/The

The great circle track was used for each leg.

For the winter season, Dec.-Feb. at 300 mb., the following values were computed from charts of mean vector wind and standard vector deviation:

$$\bar{x} = 58 \text{ kt.}, \quad \sigma_1 = 18.1; \quad \bar{y} = 45, \quad \sigma_2 = 17.4$$

Also a = 4 kt. approximately, and for h = 300 n.m., $\rho = 0.76$, from Table 1.

Table 2 gives the computed values of P(A) + P(B) for different values of r.

Table 2

r (kt.)	P(A) + P(B)	r	P(A) + P(B)	r	P(A) + P(B)
0	.00051 dr	35	.01798	70	.00969
5	.00109	40	.02134	75	.00628
10	.00215	45	.02320	80	.00375
15	.00386	50	.02320	85	.00202
20	.00648	55	.02121	90	.00101
25	.00989	60	.01782	95	.00046
30	.01392	65	.01375	100	.00019

Table 3 shows the percentage frequencies with which equivalent head-winds will be exceeded in winter at 300 mb. (a) on the shorter route only, (b) on the alternative route only, (c) using the most favourable of the two. 'Compensated' values are given in (b) and (c), so that all the winds refer to a route length of 3008 nautical miles.

Table 3

	a	b	c
100 kt.	1%	0.2%	0%
90	4	0.9	0.5
80	11	4	3
70	25	11	9
60	46	26	23
50	67	48	44
40	84	70	67
30	94	86	85
20	98	95	95
10	99.5	99	99
0	100	100	100

The frequency with which each route is the more favourable is obtained by summing the values of P(A) and P(B) separately. In this case, the alternative route is preferable on 77%, and the shorter route on 23% of occasions.

/Extension

Extension to three routes

Let x, y, z be equivalent head-winds on three tracks, means $\bar{x}$, $\bar{y}$, $\bar{z}$ and standard deviations σ_1 σ_2 σ_3 . ρ_{12} ρ_{13} ρ_{23} are the total correlations between the mean equivalent head-winds on the three routes. Then the normal trivariate distribution of x, y, z as given in standard text-books (see e.g. Aitken[4]) is

$$\phi(x,y,z) = \frac{1}{(2\pi)^{3/2}\Delta^{1/2}\sigma_1\sigma_2\sigma_3} \exp\left[-\frac{Q}{2\Delta}\right]$$

where $\Delta = 1 + 2\rho_{12}\rho_{13}\rho_{23} - \rho_{12}^2 - \rho_{13}^2 - \rho_{23}^2$

and $$Q = \frac{(x-\bar{x})^2}{\sigma_1^2}(1-\rho_{23}^2) + \frac{(y-\bar{y})^2}{\sigma_2^2}(1-\rho_{13}^2) + \frac{(z-\bar{z})^2}{\sigma_3^2}(1-\rho_{12}^2)$$
$$- \frac{2(\rho_{12}-\rho_{13}\rho_{23})(x-\bar{x})(y-\bar{y})}{\sigma_1\sigma_2} - \frac{2(\rho_{13}-\rho_{12}\rho_{23})(x-\bar{x})(z-\bar{z})}{\sigma_1\sigma_3}$$
$$- \frac{2(\rho_{23}-\rho_{12}\rho_{13})(y-\bar{y})(z-\bar{z})}{\sigma_2\sigma_3}$$

Let b and c be the reduction in equivalent head-wind necessary on the second and third routes to compensate for their additional length. Then r will lie in the range (r, r + dr) if any of the following applies:

A) $r < x < r + dr$ and $r - a < y$, $r - b < z$
B) $r - a < y < r - a + dr$ and $r < x$, $r - b < z$
C) $r - b < z < r - b + dr$ and $r < x$, $r - a < y$

Then $$P(A) = dr \int_{r-b}^{\infty}\int_{r-a}^{\infty} \phi(r,y,z)\,dy\,dz$$

$$P(B) = dr \int_{r-b}^{\infty}\int_{r}^{\infty} \phi(x,r-a,z)\,dx\,dz$$

$$P(C) = dr \int_{r-a}^{\infty}\int_{r}^{\infty} \phi(x,y,r-b)\,dx\,dy$$

Simplifying,

$$P(A) = \frac{dr}{\sigma_1\sqrt{2\pi}} \exp\left[-\frac{(r-\bar{x})^2}{2\sigma_1^2}\right] \int_{q_1}^{\infty}\int_{q_2}^{\infty} f(y,z,R_1)\,dy\,dz$$

$$P(B) = \frac{dr}{\sigma_2\sqrt{2\pi}} \exp\left[-\frac{(r-a-\bar{y})^2}{2\sigma_2^2}\right] \int_{q_3}^{\infty}\int_{q_4}^{\infty} f(x,z,R_2)\,dx\,dz$$

$$P(C) = \frac{dr}{\sigma_3\sqrt{2\pi}} \exp\left[-\frac{(r-b-\bar{z})^2}{2\sigma_3^2}\right] \int_{q_5}^{\infty}\int_{q_6}^{\infty} f(x,y,R_3)\,dx\,dy$$

/where

where $f(x, y, R) = \frac{1}{2\pi\sqrt{1-R^2}} \exp\left[-\frac{1}{2(1-R^2)}(x^2+y^2-2Rxy)\right]$,

$$R_1 = \frac{\rho_{23}-\rho_{12}\rho_{13}}{\sqrt{(1-\rho_{12}^2)(1-\rho_{13}^2)}}, \quad R_2 = \frac{\rho_{13}-\rho_{12}\rho_{23}}{\sqrt{(1-\rho_{12}^2)(1-\rho_{23}^2)}}, \quad R_3 = \frac{\rho_{12}-\rho_{13}\rho_{23}}{\sqrt{(1-\rho_{13}^2)(1-\rho_{23}^2)}}$$

and the limits of integration q are as follows:

$$q_1 = (1-\rho_{13}^2)^{-\frac{1}{2}}\left(\frac{r-b-\bar{z}}{\sigma_3} - \rho_{13}\frac{r-\bar{x}}{\sigma_1}\right); \quad q_2 = (1-\rho_{12}^2)^{-\frac{1}{2}}\left(\frac{r-a-\bar{y}}{\sigma_2} - \rho_{12}\frac{r-\bar{x}}{\sigma_1}\right)$$

$$q_3 = (1-\rho_{23}^2)^{-\frac{1}{2}}\left(\frac{r-b-\bar{z}}{\sigma_3} - \rho_{23}\frac{r-a-\bar{y}}{\sigma_2}\right); \quad q_4 = (1-\rho_{12}^2)^{-\frac{1}{2}}\left(\frac{r-\bar{x}}{\sigma_1} - \rho_{12}\frac{r-a-\bar{y}}{\sigma_2}\right)$$

$$q_5 = (1-\rho_{23}^2)^{-\frac{1}{2}}\left(\frac{r-a-\bar{y}}{\sigma_2} - \rho_{23}\frac{r-b-\bar{z}}{\sigma_3}\right); \quad q_6 = (1-\rho_{13}^2)^{-\frac{1}{2}}\left(\frac{r-\bar{x}}{\sigma_1} - \rho_{13}\frac{r-b-\bar{z}}{\sigma_3}\right)$$

Integrals of the bivariate normal distribution $\int_y^\infty \int_x^\infty f(x, y, R)\, dxdy$ have been tabulated by Pearson[5] for values of R at intervals of .05, and hence the probability that r lies in the range (r, r+dr), which equals P(A)+P(B)+P(C), can be computed for appropriate values of r.

London - New York route

The distribution of r has been computed for October at 300 mb. on the London - New York route, using the two previous tracks, and one further north. The three tracks are: (Fig. 2)

1. London - Fastnet - Torbay - New York (3008 nautical miles)
2. London - Malin - Goose Bay - New York (3055 nautical miles)
3. London - Prestwick - 60½ N 50 W - New York (3198 nautical miles).

As before, great circle routes are used on each individual leg. From seasonal charts

$$\bar{x} = 44, \quad \sigma_1 = 17.3; \quad \bar{y} = 35, \quad \sigma_2 = 17.2; \quad \bar{z} = 29, \quad \sigma_3 = 15.9$$

these corresponding to routes 1, 2, and 3 respectively.

Taking the separation 1 to 2 as 300 n.m., 2 to 3 as 150 and 1 to 3 as 450,

$$\rho_{12} = 0.76, \quad \rho_{23} = 0.87, \quad \rho_{13} = 0.60.$$

For ease in computation these values have been very slightly adjusted, and give $R_1 = +0.80$, $R_2 = -0.20$, $R_3 = +0.60$. Also the winds to compensate for the additional route lengths are

$$a = 4 \text{ kt.}, \qquad b = 16 \text{ kt.}$$

Table 4 gives computed values of P(A) + P(B) + P(C) for different values of r.

/Table 4

Table 4

r (kt.)	P(A)+P(B)+P(C)	r	P(A)+P(B)+P(C)	r	P(A)+P(B)+P(C)	r	P(A)+P(B)+P(C)
-20	.00008 dr	10	.00715	40	.02388	70	.00242
-15	.00021	15	.01101	45	.02095	75	.00116
-10	.00052	20	.01549	50	.01667	80	.00050
-5	.00115	25	.01987	55	.01197	85	.00019
0	.00230	30	.02320	60	.00784	90	.00007
5	.00425	35	.02468	65	.00459	95	.00002

Table 5 gives the percentage frequencies with which winds at 300 mb. in October would be exceeded on tracks 1, 2 and 3 alone, and on the most favourable of the three, all winds referring to a route length of 3008 n.m.

Table 5

r (kt.)	Track 1	Track 2	Track 3	Most favourable
100	0.1%	0%	0%	0%
90	0.4	0.2	0.2	0
80	2	1	1	0.2
70	7	4	6	1.5
60	18	11	17	6
50	36	26	38	18
40	59	48	62	39
30	79	70	83	63
20	92	87	94	83
10	98	95	99	94
0	99.5	99	99.8	98
-10	99.9	99.8	100	99.7
-20	100	100	100	100

When the most favourable route is chosen, track 1 would be flown on 31% of occasions, track 2 on 50%, and track 3 on 19%.

Conclusion

Using assumptions similar to those at present made in computing statistics of equivalent head-wind over single tracks, the corresponding frequency distribution when the most favourable of two or three tracks is flown may be obtained. The examples illustrate the order of magnitude of reduction in head-wind that may be achieved. For three routes whose mean compensated equivalent head-wind is similar, the reduction is considerable, especially at the higher velocities. The variable track technique requires, of course, accurate wind prediction; errors in the latter have not been considered in this paper.

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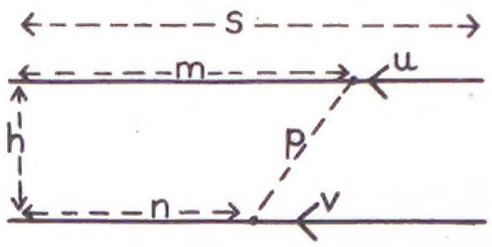

Figure 1.

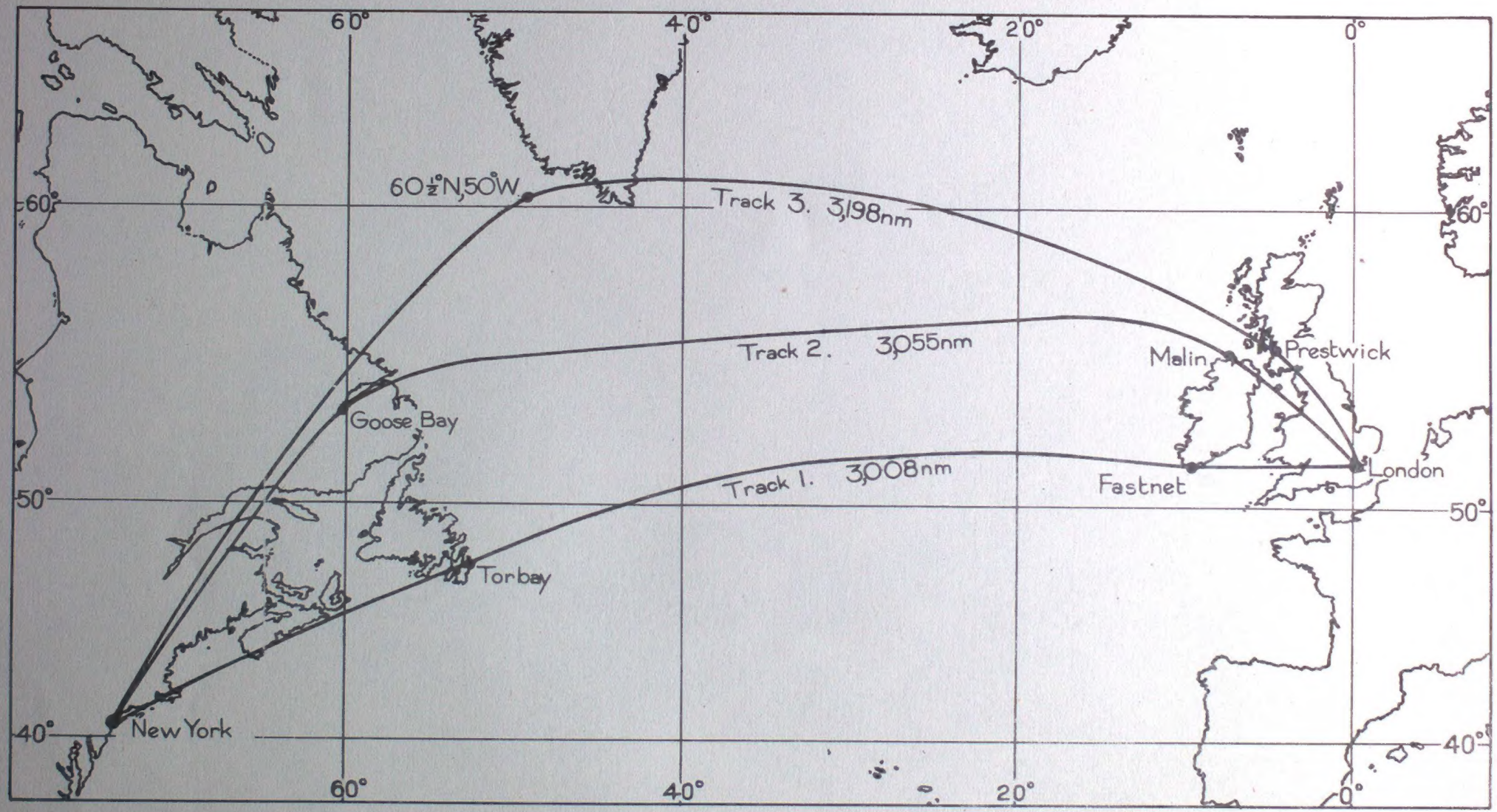

Figure 2.

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